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Lifshitz' Law for the Volume of a Two-Dimensional Droplet at Zero Temperature

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We study a simple model of the zero-temperature stochastic dynamics for interfaces in two dimensions—essentially Glauber dynamics of the two-dimensional Ising model at $T=0$. Using elementary geometric considerations, we show that the (rescaled) volume of an initially square droplet decreases linearly to zero as a function of (rescaled) time.

KEY WORDS: Lifshitz' law; droplets; 2D interfaces.

1. INTRODUCTORY REMARKS

In the early 1960s, Lifshitz[5] proposed that, for a multiple-phase system which has a droplet of one phase immersed in another, the droplet radius R obeys the dynamical equation

$$\frac{dR}{dt} = -\frac{[\text{const}]}{R} \tag{1a}$$

For two dimensions, assuming the droplet to be roughly spherical, (1a) may be expressed in terms of the volume V of the droplet as

$$\frac{dV}{dt} = -[\text{const}] \tag{1b}$$

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In the ensuing years, there has not been any serious doubt about the validity of (1)—at least for uniform systems. (For disordered systems, it is anticipated that the above will break down due to the "pinning of interfaces."[2]) Recently, there have been some results that have pushed toward a rigorous understanding of Lifshitz' picture—at least in certain extreme cases. All of these, in one form or another, involve hydrodynamic limits. Thus, for a droplet of actual volume $V \sim N^2$, the time is rescaled by N^2 and one investigates, e.g., the limiting (stochastic) behavior of the quantity $v(t) = (1/N^2)\, V(N^2 t)$.

Foremost is the recent result of ref. 3. Here the Ising model is investigated in the limit of infinite range of coupling. In this limit, there is no sharp domain wall separating regions of opposite spin type. However, the boundary of a region may be defined via level curves and, in this sense, the laws (1a) and (1b) can be demonstrated. Next, there is the work of Spohn[8] (see also ref. 7 and the derivation in ref. 6), where the zero-temperature dynamics of interfaces for nearest-neighbor Ising systems was shown, in the hydrodynamic limit, to follow motion by (modified) mean curvature. Such a rule for interface motion is the essence of the original derivation by Lifshitz. However, the setup required in ref. 8 was an infinitely tall cylinder, of girth N, divided into two infinite components ("top" and "bottom") separated by a single interface. It was further assumed that the interface could be expressed as the graph of a function. Hence, in the hydrodynamic ($N \to \infty$) limit, the long-time behavior results in the relaxation of the interface into a straight line. Evidently the work in ref. 8 does not address the issue of a droplet of one phase *immersed* in another. (Purportedly, the techniques of ref. 8 can be easily extended to handle these latter cases, but the authors of this note have no concept of how to go about making such an extension.) Finally, in the context of hydrodynamic limits for 1D particle systems with moving boundaries (Stefan's problem) a Lifshitz law—along with a shape reconstruction theorem—can be surmised for the case of an (immersed) droplet that is trapped against the corner of an infinite quadrant.[1]

Here we will present a straightfoward derivation of Lifshitz' law, in the form of Eq. (1b), for a simple model of stochastic dynamics of two-dimensional interfaces. However, we will not address the delicate (and important) issue of the limiting dynamics for the motion of the interface itself. Nevertheless, the fothcoming has a certain appeal due its simplicity. Furthermore, it is distinguished from much of the above mentioned because it deals directly with the interface rather than an underlying particle system.

2. DEFINITION OF THE MODEL

The interface in our model is defined as a closed, self-avoiding path along the bonds of $\mathbb{Z}^2$. The interface configuration at time t will be denoted by $I(t)$ and its length by $|I(t)|$. This interface may be envisioned as separating regions of Ising spins—here sitting on the dual lattice—that are of opposite type. Thus, in the case of a "droplet," the minority species is a simply connected cluster and $I(t)$ is its boundary. For the purposes of this paper, the initial condition will always be rectangular. Let x be a dual site and let i_x denote the four bonds that surround x. The sites of interest are those x's for which i_x meets I, i.e., those sites in the cluster with a neighbor outside the cluster or vice versa. Then, formally, we can add or delete x from the cluster by saying that

$$I \to I \circ \mathrm{i}_x \tag{2}$$

where $I \circ \mathrm{i}_x = I \cup \mathrm{i}_x \backslash I \cap \mathrm{i}_x$ is the symmetric difference operator. The dynamics is defined by the rules that for all x (of interest),

$$I \to I \circ \mathrm{i}_x \quad \text{at rate} \quad \begin{cases} 0 & \text{if} \quad |I \circ \mathrm{i}_x| > |I| \\ 1 & \text{if} \quad |I \circ \mathrm{i}_x| = |I| \\ 2 & \text{if} \quad |I \circ \mathrm{i}_x| = |I| - 2 \\ 4 & \text{if} \quad |I \circ \mathrm{i}_x| = |I| - 4 \end{cases} \tag{3a}$$

subject to the restriction that (for $I \neq \mathrm{i}_x$)

$$I \circ \mathrm{i}_x \quad \text{is a simple closed curve} \tag{3b}$$

The above constitutes a simple model of stochastic interfacial dynamics that is governed by local rules. The various allowed and forbidden moves are illustrated in Fig. 1.

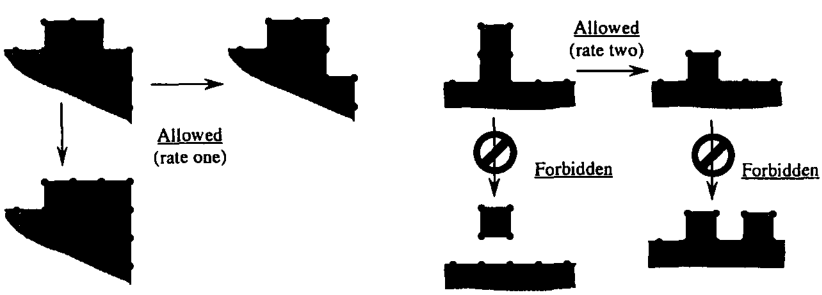

Fig. 1

Remark. The rate for the transition where I reduces by 4 is sheer self-indulgence, since in any realization, there is only one such move and this move marks the end of the process. The restriction (3b) sounds serious, but is probably not of any real significance for the initial configurations that we will consider. In general, this restriction forbids the fissioning of droplets. If the initial condition is a rectangle, it turns out that under the dynamics described in Eq. (3a), the only mechanism for splitting is when a finger of width one gets sliced off. Such fingers are probably rare enough to begin with and, evidently, are extremely short-lived. Relaxation of the rule (3b) would serve to further shorten the lifetime of a finger; however, it would also cause spurious complications, and hence lengthen our derivation. It is seen that, aside from this restriction (and the defined rate at the terminus), the model we are studying is a standard version of Glauber dynamics (know as Gibbs sampler) with a flip rate of 2, at zero temperature.

3. RESULTS AND DERIVATIONS

We start with the following observation: Traversing the curve $I(t)$ in the *positive* (counterclockwise) direction, let $P(t)$ denote the number of left-handed turns and $Q(t)$ the number of right-hand turns. Since $I(t)$ is, after all, a simple closed curve, it follows that the total rotation of the tangent is 2π. Since each left-handed turn is an increment of $\pi/4$ and each right-handed turn $-\pi/4$, it follows that for any time $|I(t)| > 0$,

$$P(t) - Q(t) = 4 \tag{4}$$

The above leads immediately to the following:

Proposition 1. Let $V(t)$ denote the dynamic volume (area) of the interface $I(t)$. Then, provided that the process has not terminated, the average rate of change of the volume is a constant. Explicitly:

$$\left.\frac{d}{ds}\mathbf{E}(V(t+s) \mid I(t);\, V(t) > 0)\right|_{s=0}$$
$$\equiv \lim_{\delta t \to 0} \frac{E(V(t+\delta t) \mid I(t);\, V(t) > 0) - V(t)}{\delta t}$$

is exactly equal to -4.

Proof. The above is obvious if $|I| = 4$, so let us assume otherwise. Moving around the curve in a positive direction, we claim that each left-handed turn that is not immediately preceded by or followed by another left-handed turn represents an "outward-pointing corner" that could be potentially lost in an $|I|$-preserving move. These moves will then decrease the volume $V(t)$ by one and happen at rate one. The above sentence can be checked, locally, by exhausting all 4 (or, to be precise, 16) cases. Assume, without loss of generality, that we start walking vertically up the page from the site $s_1 \to s_2$ and then we move horizontally from s_2 to s_3 as illustrated in Fig. 2. Accoding to the stipulation that we have not immediately come out of nor will immediately go into another left-handed turn, the possible candidates for the sites preceding and following the path (s_1, s_2, s_3) are, respectively, s_0 or s_0' and s_4 or s_4' as illustrated. Since the droplet region (shaded) by definition lies to the left of (s_1, s_2) the validity of the above statement is manifest.

Similarly, if there are two successive left turns, this corresponds to a "protuberance" which permits an $|I| \to |I| - 2$ move and these happen at rate 2. By this convention, it follows that each left-handed turn contributes equally to the erosion of the volume. A similar argument shows that each right-handed turn contributes equally to the possibility of increasing the volume of the droplet and, clearly, a region with no turns remains just that. Evidently, we have shown

$$\mathrm{Prob}(V(t+\delta t) = V(t) + 1 \mid I(t);\ V(t) > 0) = P(t)\,\delta t + O([\delta t]^2) \quad (5a)$$

and

$$\mathrm{Prob}(V(t+\delta t) = V(t) - 1 \mid I(t);\ V(t) > 0) = Q(t)\,\delta t + O([\delta t]^2) \quad (5b)$$

from which the result follows.

Remark. The preceding is already a (very) weak form of the Lifshitz law in the sense that it indicates a linear loss of volume with time. There are, however, two major issues that must be dealt with: First there is the

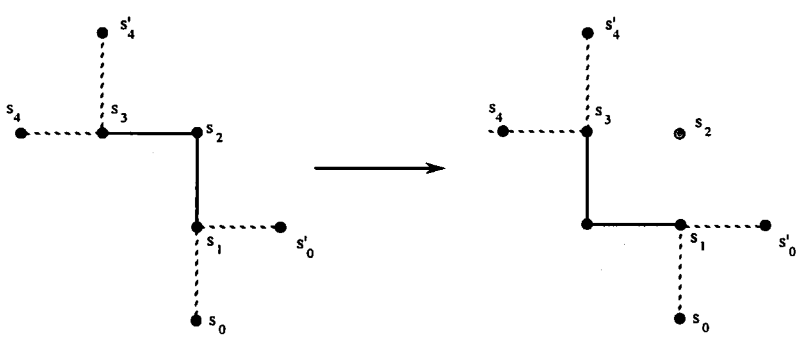

Fig. 2

fact that the statement in Proposition 1 pertains to the derivative of a *conditional* probability; we must remove this conditioning. Second, we must control the fluctuations. For example, if we consider a process X_t on $\mathbb{Z}^+$ with transition rates $X_t \to X_t + 1$ at rate $X_t + 1$ and $X_t \to [X_t - 1]^+$ at rate X_t, it is easy to show that $\mathbf{E}(X_t) \sim t$. However, it is also the case that $\mathrm{Var}(X_t) \sim t^2$, so in this example the statement $X_t \sim t$ would be unrealistic. In our case, we can show that the standard deviation is small relative to the mean and thus a meaningful Lifshitz law will be established.

Theorem 1. Let $I_N(t)$ denote the interface at time t corresponding to an initial configuration $I(0)$ that is an $N \times \lambda N$ rectangle. Let $V_N(t)$ denote the dynamic volume and $v_N(t)$ the rescaled volume as a function of rescaled time:

$$v_N(t) = \frac{V_N(N^2 t)}{V_N(0)}$$

Then, for any $t < \infty$, $v_N(t) \to v(t)$, in L^2, where

$$v(t) = \begin{cases} 1 - 4t; & t \leqslant 1/4 \\ 0; & t \geqslant 1/4 \end{cases}$$

Proof. It is more convenient to deal with the eroded volume

$$\Delta_N(t) = V_N(0) - V_N(t) \tag{6}$$

The result of the previous proposition amounts to the statement that for all t

$$\frac{d}{ds} \mathbf{E}(\Delta_N(t+s) \mid I_N(t); \Delta_N(t) < V_N(0)) \bigg|_{s=0} = 4 \tag{7}$$

and an easy calculation based on Eqs. (5a) and (5b) shows that

$$\frac{d}{ds} \mathbf{E}(\Delta_N^2(t+s) \mid I_N(t); \Delta_N(t) < V_N(0)) \bigg|_{s=0} = 8\Delta_N(t) + [P_N(t) + Q_N(t)] \tag{8}$$

Equation (8) is clearly the key: neglecting again the problem of conditioning, the first term is exactly what is expected if $\Delta(t) \approx 4t$ and, obviously, one anticipates that $\Delta \gg P + Q$.

We will now dispense with the annoyance caused by the conditioning. Let τ_N denote the stopping time defined by

$$\tau_N = \sup\{t \mid \Delta_N(t) < V_N(0)\} \tag{9}$$

and let us define

$$D_N(t)=\begin{cases}\Delta_N(t); & t<\tau_N\\ V_N(0)+4(t-\tau_N); & t>\tau_N\end{cases}\tag{10}$$

We see that Eqs. (7) and (8) can be replaced with the more palatable

$$\frac{d}{ds}\mathbf{E}(D_N(t+s)\mid I_N(t))\Big|_{s=0}=4\tag{11a}$$

(i.e., $D_N(t)-4t$ is a martingale) and

$$\frac{d}{ds}\mathbf{E}(D_N^2(t+s)\mid I_N(t))\Big|_{s=0}=8D_N(t)+[P_N(t)+Q_N(t)]\tag{11b}$$

where, of course, $P_N(t)+Q_N(t)\equiv 0$ when $t>\tau_N$. Note that we can revert back to the original variables by the identity

$$V_N(t)=[V_N(0)-D_N(t)]^+\tag{12a}$$

or, defining $d_N(t)=D_N(t)/V_N(0)$,

$$v_N(t)=[1-d_N(t)]^+\tag{12b}$$

Henceforth, we will work with the d-type variables.

We now claim that in every configuration,

$$P_N(t)+Q_N(t)\leqslant 4\{[2\Delta_N(t)]^{1/2}+1\}\leqslant 4\{[2D_N(t)]^{1/2}+1\}\tag{13}$$

Indeed, examining Fig. 3, we may define the current box containing the droplet by the extreme points on the right side, R and R', the left, L and L', etc. We make a few elementary observations which, due to the initial condition, are true at $t=0$ and thereafter are dynamically enforced.

1. All bonds in the line segment RR' are in I.

2–4. Similarly for LL', BB', and TT'.

5. In portion I between T and R', the P- and Q-type corners alternate starting and ending with P-type corners at T and R' (i.e., in this region, I can be represented as the graph of a function).

6–8. Similarly for B' and R, L' and B, and T' and L.

Properties 1–8 are easily checked.

We denote by $Q_N^{[LB]},\ldots, Q_N^{[TL]}$ the number of Q-type corners in the respective portions of I_N between L' and $B,\ldots, T'$ and L, and by

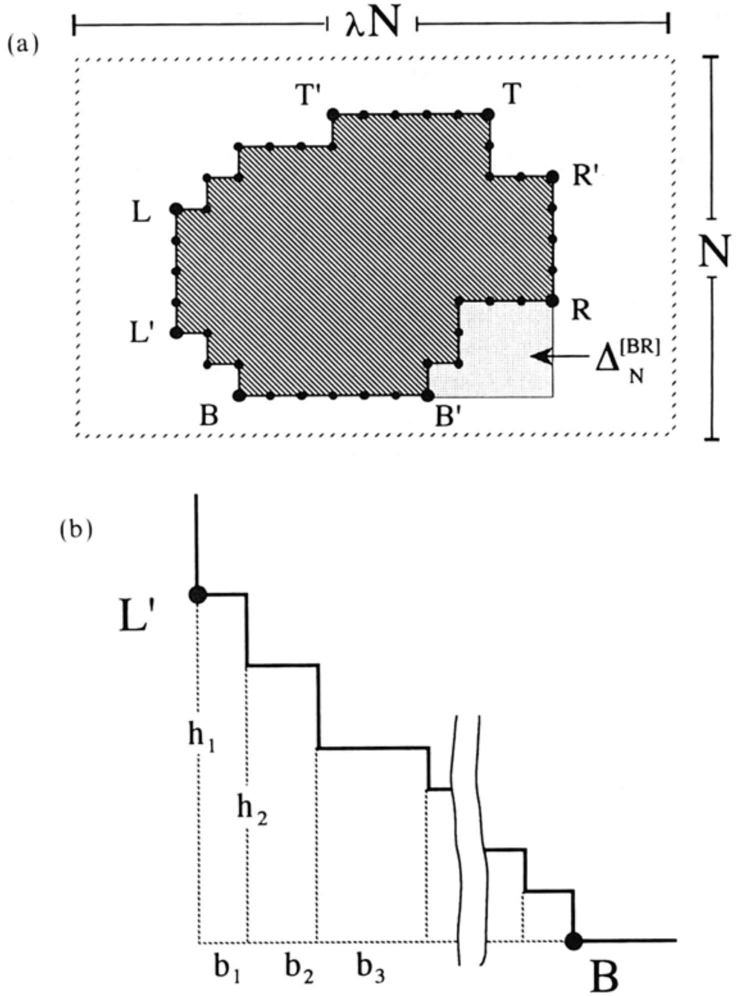

Fig. 3

$\Delta_N^{[LB]},\ldots,\Delta_N^{[TL]}$ the amount that has "eroded" from each of these four corners (see Fig. 3a). Let us examine, e.g., the lower left corner region (between L' and B; see Fig. 3b). As is clear from the figure,

$$\Delta_N^{[LB]} = b_1 h_1 + b_2 h_2 + \cdots + b_{Q_N^{[LB]}} h_{Q_N^{[LB]}} \tag{14}$$

where the b's are the distances between successive P-type corners and the h's represent the corresponding vertical drops.

Since each $b_k \geqslant 1$ and each $h_k \geqslant h_{k+1} + 1$, we see that

$$\Delta_N^{[LB]} \geqslant \tfrac{1}{2}(Q_N^{[LB]})(Q_N^{[LB]} + 1) \tag{15}$$

Thus, collecting the contributions from al four corners, and noting that $2(\Delta_N)^{1/2} \geqslant (\Delta_N^{[LB]})^{1/2} + \cdots + (\Delta_N^{[TL]})^{1/2}$, we arrive at

$$4(2\Delta_N)^{1/2} + 4 \geqslant Q_N + P_N \tag{16}$$

as promised.

Substituting the bound in Eq. (16) into Eq. (11b) and integrating from t to $t + \delta t$, we get

$$\mathbf{E}(D_N^2(t+\delta t)) - D_N^2(t) \leqslant \{8D_N(t) + 4[2D_N(t)]^{1/2} + 4\}\ \delta t + O(|\delta t|^2) \tag{17}$$

(Notice that due to the use of the D_N instead of the Δ_N there is no issue with the condition $t \geqslant \tau_N$ versus $t \leqslant \tau_N$—although the right-hand side of Eq. (17) is still random.) Averaging Eq. (17) up to time t and then dividing by δt, we obtain

$$\begin{aligned} \frac{d}{dt}\mathbf{E}(D_N^2(t)) &\leqslant 8\mathbf{E}(D_N(t)) + 4\mathbf{E}[2D_N(t)]^{1/2} + 4 \\ &\leqslant (8)(4)t + c_1\sqrt{t} + c_2 \end{aligned} \tag{18}$$

where c_1 and c_2 are constants of order unity. Integrating Eq. (18), we find

$$\mathbf{E}[D_N^2(t)] - [\mathbf{E}D_N(t)]^2 \leqslant ct^{3/2} \tag{19}$$

where c is a constant of order unity. Thus we obtain, for any $t < \infty$,

$$d_N(t) \to 4t \tag{20}$$

in L^2. The stated result follows from Eq. (12b). ∎

Our results can now be summarized by the following statement:

Theorem 2. Let $v_N(t)$ and $v(t)$ be as described in the statement of Theorem 1, with the $v_N(t)$ taken to be right continuous. Then $\sup_t |v_N(t) - v(t)| \to 0$ in probability. Explicitly, for any ε,

$$\text{Prob}[\sup_t |v_N(t) - v(t)| > \varepsilon] \leqslant \frac{\tilde{c}}{\varepsilon^2 N}$$

with $\tilde{c}$ a constant of the order of unity. Furthermore, $\tau_N \to 1/4$, in probability, with a similar estimate on the rate of convergence.

Proof. As mentioned previously, $D_N(t)-4t$ is a martingale and hence so is $d_N(t)-4t$. Consider the process on $[0, 1]$—where we note that $t=1$ is well past the anticipated stopping time—and let $\varepsilon \ll 1$. For discrete times $t_1 ..., t_k$, $t_0=0$, $t_k=1$, Doob's inequality gives us

$$\text{Prob}[\max_{j \leqslant k} |d_N(t_j)-4t| > \varepsilon] \leqslant \frac{1}{\varepsilon^2}[\mathbf{E}(d_N(t_k)-4t_k)^2] \leqslant \frac{\tilde{c}}{\varepsilon^2 N} \tag{21}$$

where $\tilde{c}$ is a constant of the order of unity.

By a straightforward argument, (see, e.g., ref. 4, 244), Eq. (21) can be extended to any countable dense subset of $[0, 1]$. Since, with probability one, $d_N(t)$ has only a finite number of jumps on any finite interval, this implies

$$\text{Prob}[\sup_{t \leqslant 1} |d_N(t)-4t| > \varepsilon] \leqslant \frac{\tilde{c}}{\varepsilon^2 N} \tag{22}$$

Thus, if N is large, $d_N(t)$ is confined to a small window, as ilustrated in Fig. 4. Moreover, under these circumstances, at $t=1$, d_N is certainty greater than one and we are well past the stopping time τ_N: once past this stopping

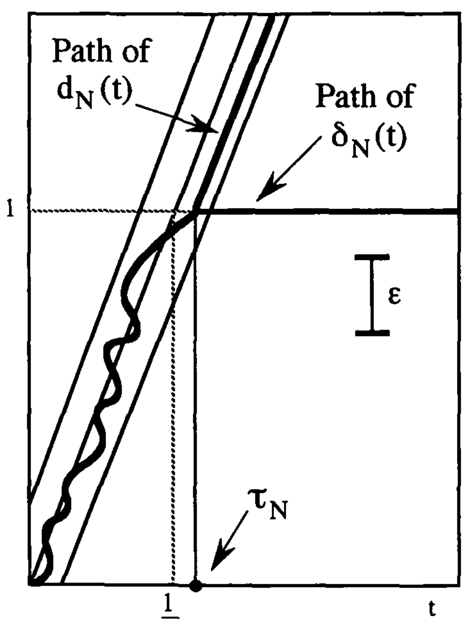

Fig. 4

time the difference $d_N(t) - 4t$ is a constant, here bounded by ε. Hence Eq. (22) may be replaced with

$$\mathrm{Prob}[\sup_t |d_N(t) - 4t| > \varepsilon] \leqslant \frac{\tilde{c}}{\varepsilon^2 N} \tag{23}$$

Comparing the paths of $d_N(t)$ and $\delta_N(t) \equiv \Delta_N(N^2 t)/V_N(0)$, it is clear that the above implies $\delta_N(t) \to \max\{4t, 1\}$ and $\tau_N \to 1/4$, in probability, with similar estimates on the rates of convergence. ∎

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